VOLUME 81 PAPER No. 811

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

OCTOBER, 1955

A GENERAL METHOD OF ANALYSIS OF RIGID FRAMES

by Panagiotis D. Moliotis, M. ASCE

STRUCTURAL DIVISION

{Discussion open until February 1, 1956}

Copyright 1955 by the AMERICAN SOCIETY OF CIVIL ENGINEERS
Printed in the United States of America

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

THIS PAPER

--represents an effort by the Society to deliver technical data direct from the author to the reader with the greatest possible speed. To this end, it has had none of the usual editing required in more formal publication procedures.

Readers are invited to submit discussion applying to current papers. For this paper the final date on which a discussion should reach the Manager of Technical Publications appears on the front cover.

Those who are planning papers or discussions for "Proceedings" will expedite Division and Committee action measurably by first studying "Publication Procedure for Technical Papers" (Proceedings Paper No. 290). For free copies of this Paper—describing style, content, and format—address the Manager, Technical Publications, ASCE.

Reprints from this publication may be made on condition that the full title of paper, name of author, page reference, and date of publication by the Society are given.

The Society is not responsible for any statement made or opinion expressed in its publications.

This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

A GENERAL METHOD OF ANALYSIS OF RIGID FRAMES*

Panagiotis D. Moliotis,[1] M. ASCE

INTRODUCTION

1.—The rigid frames to be analysed have straight members with constant moment of inertia.

The members are numbered 1,2,3, ... and the joints A,B,C, ... The bending moments , normal forces N, shear forces Q, the end section of a member i adjoining a joint K are named M_{ki}, N_{ki}, Q_{ki}, respectively.

The above moments, normal forces and shear forces when considered as exerted by the member i to the joint K, the joint assumed to be fully restrained against rotation and displacement, are called "on the joint" moments, normal forces and shear forces from the member i to the joint K and they are denoted by the symbols $\overline{M}_{ki}$, $\overline{N}_{ki}$, and $\overline{Q}_{ki}$.

Evidently the on the joint moments are equal but opposite in sign to the fixed end moments of the members. The sum of the moments and forces exerted to the joint K by all the members meeting at it is called the "unbalanced moment" $\Sigma\overline{M}_{ki}$ or normal force $\Sigma\overline{N}_{ki}$ or shear force $\Sigma\overline{Q}_{ki}$ at the joint K.

As it is known loaded structures undergo elastic deformations, main characteristics of which are the movements of the joints. These latter fall into the following two categories:

1) Rotations and 2) Displacements.

In rigid frames with straight members the deformations caused by normal forces and shear forces are negligible compared with the deformations caused by bending moments and they are neglected.

Due to the deformations caused by the bending moments the resulting displacements of the joints are interrelated and so the displacements of each joint are not independent from the displacement of the other joints.

With the aid of diagrams of virtual displacements the displacements of all the joints can be expressed as functions of the displacements of a certain number of them.

The number of these fundamental joints gives the degrees of elastic freedom of the frame.

If we replace the joints by hinges we can convert a rigid frame into a formation which, as a rule, possesses a certain number of degrees of geometrical freedom. We can eliminate these degrees of geometrical freedom by adding a certain number of imaginary simple restraints by means of rollers.

The degrees of geometrical freedom of the above pin-joined formation are the same as the degrees of elastic freedom of the original rigid frame.

The number of the roller restraints needed to be added to the above not rigid pin-joined formation to convert it into a rigid one gives again, in

*Originally published, in Greek, in Technika Chronika in 1952.

1. Asst. Prof., Civ. Eng., National Technical Univ., Athens, Greece.

accordance with the afore mentioned, the degrees of the elastic freedom of the original rigid frame.

Thus by adding one or two or more imaginary roller restraints we can reduce by equal numbers the degrees of elastic freedom of the rigid frame. The introduced imaginary roller restraints are numbered I, II, III,. . .

2.—We call "group of condisplaced joints" corresponding to a certain joint, all the joints of the frame which will be displaced together with it should the roller restraint at it be removed.

We call "group of condisplaced members," corresponding to a certain joint, all the members of the frame which will be displaced together with it should the roller restraint at it be removed.

It is possible that joints and members of the frame belong to two or more groups of condisplaced joints or members.

The following symbols will be used:

δ_{MI}: for the displacements of the joints of the frame. The suffix M refers to the joint and the suffix I to the group of the condisplaced joints to which the considered joint M belongs.

δ_{iI}: for the displacements of the ends of a member i relatively to each other perpendicular to the direction of the axes of the member. The suffix i refers to the member and the suffix I to the group of the condisplaced members to which the considered member belongs.

η_{iI}: for the component of displacement of member parallel to its axis. The suffixes i and I have the same meaning as above.

Convention of Signs

A bending moment exerted by a member to a joint is positive when clockwise. Conversely the bending moment at the end sections of a member are positive when anticlockwise (Fig. 1a).

A shear force is positive when an infinite portion of the member tends to turn clockwise under the action of the shear forces at its ends (Fig. 1b).

A normal force is positive when causes tension to the member as in Fig. 1c.

The angles of rotation of the joints are positive when clockwise.

The displacements δ_i and η_i are positive when they are obtained by a clockwise rotation of a member i around the instantaneous center O_i of the member (Fig. 2).

Stress Analysis of the Frame in Two Steps

3.—Let us consider the frame of the Fig. 3 loaded by the external loads (P). As it is known we can accomplish the stress analysis of the frame in two steps. 1) First step: Calculation of the on the joint moments produced by external loads on the members. 2) Second step: Calculation of the end bending moments and normal and shear forces of the member due to the unbalanced moments $\bar{M}_D$, $\bar{M}_E$ . . . . . and forces Z_I°, Z_H° . . . . . (Fig. 4)

The moments and forces developed at this step are in the following denoted in general by $(\phi_\circ)$.

The moments $\bar{M}_D$, $\bar{M}_E$ . . . . . are given by the following equation

$$\bar{M}_K = \sum_i \bar{M}_{Ki} + M_K \qquad (1)$$

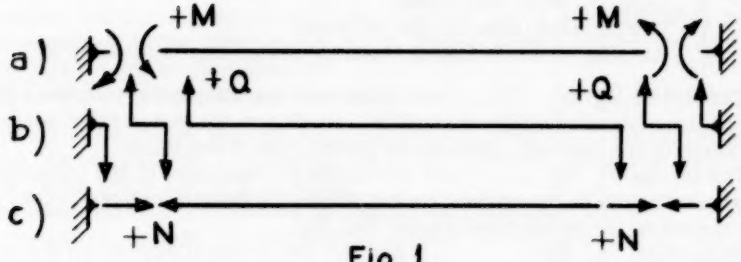

Fig. 1

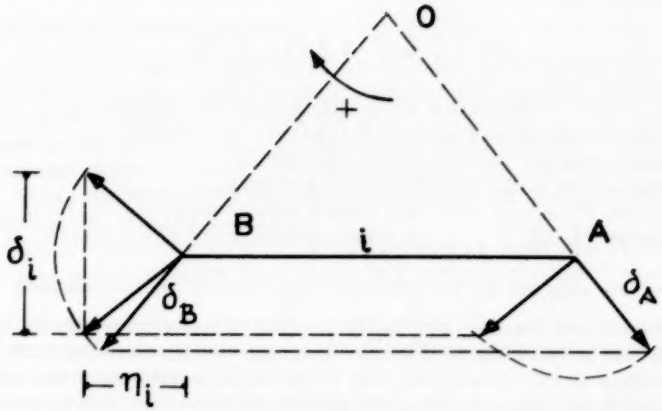

Fig. 2

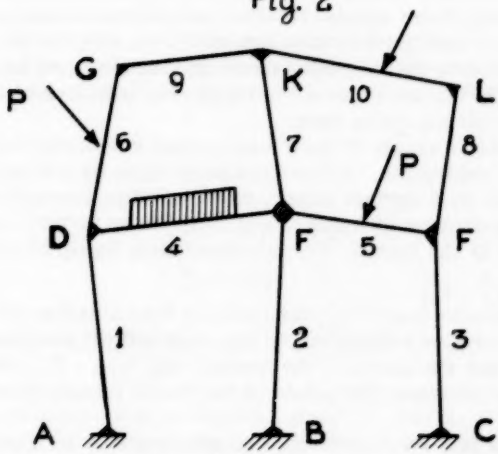

Fig. 3

where M_k is the unbalanced moment at the joint
M_k is an external moment applied on the joint
K (positive when clockwise).
i is a member meeting the joint and
$\sum_i$ the sum of the moments from all the members meeting K.

The forces Z_I°, Z_{II}° . . . . . are equal and opposite to the reactions F_I°, F_{II}° . . . of the imaginary roller restraints, which are put to a certain number of the joints of the frame to prevent displacements of the joints.

The forces F_I°, F_{II}° . . . . . are calculated as reactions of the imaginary roller restraints to the corresponding statically determined pin-joined formation loaded by the unbalanced forces $\bar{P}_D$, $\bar{P}_E$ . .

The forces $\bar{P}_D$, $\bar{P}_E$ . . . . . are the vector resultants of all the normal and shear forces and Q_{ki} (exerted by the members to the joint K) and of any external force applied to the joint.

$$\bar{P}_K = \sum_i \bar{Q}_{Ki} + \sum_i \bar{N}_{Ki} + P_K \quad \ldots\ldots\ldots \qquad (2)$$

where $\bar{Q}_{ki}$ is the on the joint K shear force from the member i.
$\bar{N}_{ki}$ is the on the joint K normal force from the member i.
P_k is any external load applied on the joint K.
The summation at the equation (2) is vectorial.
The forces F_I°, K_{II}° . . . . . are called fixing forces.
The forces Z_I°, Z_{II}° . . . . . are called displacers of the frame.

Solution of the step (ϕ°) by the Method of Successive Approximations

4.—The solution of the step (ϕ°) by the method of successive approximations is carried out in two stages. Firstly by successively unlocking each joint as far as rotation is concerned, the rest of the joints remaining fixed against rotation and displacement. Secondly by setting free each joint in turn to linear displacement by removing successfully each roller restraint, the rest of the joints remaining fixed against rotation and displacement.

When we unlock each joint to free rotation, then with the aid of distribution and carry over factors and the unbalanced moments we can calculate the bending moments at the end sections of the members, both close to the considered joint and close to all adjoining ones.

The setting free to rotate of each joint causes new fixing forces F on the imaginary roller restraints. We can calculate them by multiplying the unbalanced moments by a certain factor, corresponding to each imaginary roller restraint, the dimensions of which are (L^{-1}). These factors are called restraining factors of the frame. The corresponding fixing forces are denoted by the symbols F_I^A, F_{II}^A . . .

After the first unlocking of all the joints to free rotation we remove in turn all the imaginary roller restraints; in this removal all the joints of the frame are displaced under the action of the forces: Z_I, Z_{II} . . . . . We assume that during these displacements the joints of the frame remain fixed against rotation.

Let us assume that we remove the imaginary roller I. The force Z_I, acting on the joint where the roller I was removed, will be given by

$$Z_I = -F_I = -(F_I^\circ + F_I^A + F_I^B + \ldots\ldots\ldots)$$

Because of the displacement of each joint at the end sections of the members meeting the considered joint bending moments are developed which can be calculated by multiplying the force Z_I by a certain factor corresponding to each member and having dimensions (L^{+1}). These factors are called distribution factors due to displacement.

On the remaining imaginary roller restraints II, III . . . . . ., because of the removal of the roller Z_I, new fixing forces F^I_{II}, F^I_{III}, ... are developed which we can calculate by multiplying the Z_I by a certain numerical factor, which is called shear carry over factor.

In the same manner we remove in turn all the imaginary rollers.

We repeat the successive unlocking to rotation and setting free to displacement of the joints of the frame until we obtain unbalanced moments and forces Z negligible.

Diagram of Virtual Displacements

The calculation of the restraining factors, distribution factors due to displacement and shear carry over factors, is given in the following by the application of the principle of virtual works.

We adopt as virtual deformations the infinite displacements of the joints and bars produced by the successive removal of the imaginary roller restraints. These displacements are obtained with the aid of diagrams of virtual displacements (Fig. 5)

In the figure 5 diagram of virtual displacements of the frame is drawn, resulting from a displacement δ_I of the joint D where the roller I was removed.

In the following we shall use the symbols (η^I), (η^{II}), ... to denote diagrams of the virtual displacements obtained by the successive removal of each of the roller restraints.

We calculate the displacements of the joints and the relative displacements of the ends of the members as functions of the displacements δ_I, δ_{II}, ... as follows: (Fig. 6)

From the fig. 6 with the aid of the instantaneous center of rotation and the angles ϕ_{Ai}, ϕ_{Bi}, if we know the displacement δ_A, of the end A, we can calculate the remaining displacements of the member by using the following equations:

$$\begin{aligned} \delta_B &= C'_B \quad \delta_A \\ \delta_i &= c'_i \quad \delta_A \\ \eta_i &= d_i \quad \delta_A \end{aligned} \tag{3}$$

where

$$C'_B = \frac{\sin \varphi_{Ai}}{\sin \varphi_{Bi}}, \quad C'_i = \frac{\sin (\varphi_{Ai} + \varphi_{Bi})}{\sin \varphi_{Bi}}, \quad d'_i = \sin \varphi_{Ai}$$

For the case of a series of members 1.2 . . . . . r (Fig. 7) we can calculate the displacement of the joint and the relative displacements of the member with the aid of the equations (3) as follows:

$$\begin{aligned} &\delta_M = C'_B \ldots\ldots\ldots C'_M \; \delta_A = C'_M \; \delta_A \\ &\delta_M = C_N \; \delta_A \,, \quad \delta_r = C_r \; \delta_A \,, \quad \eta_r = d_r \; \delta_A \end{aligned} \tag{4}$$

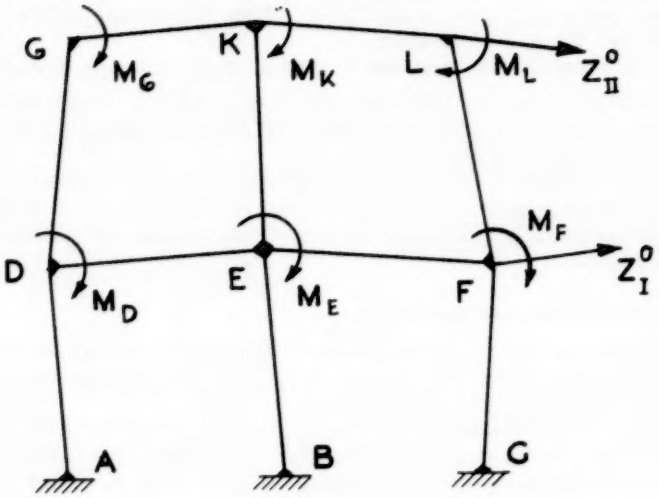
G
K
L
M_G
M_K
M_L
Z_{II}^0
D
E
F
M_D
M_E
M_F
Z_I^0
A
B
C

Fig. 4

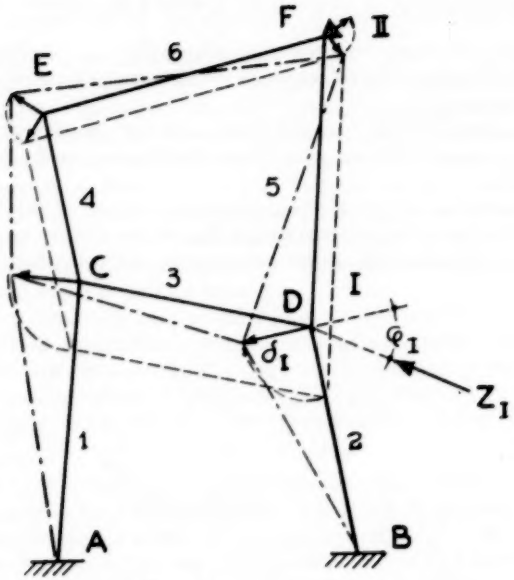
F
II
E
6
4
5
C
3
D
I
φ_I
δ_I
Z_I
1
2
A
B

Fig. 5

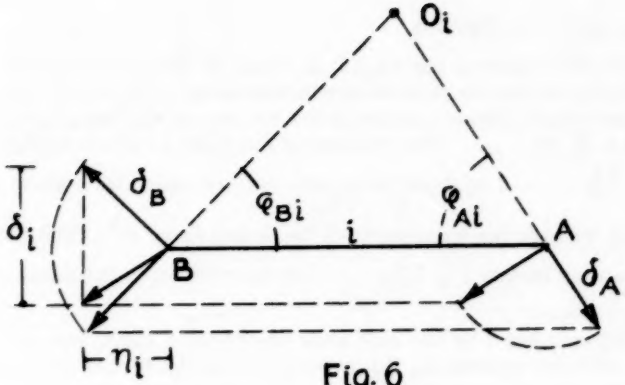

Fig. 6

Fig. 7

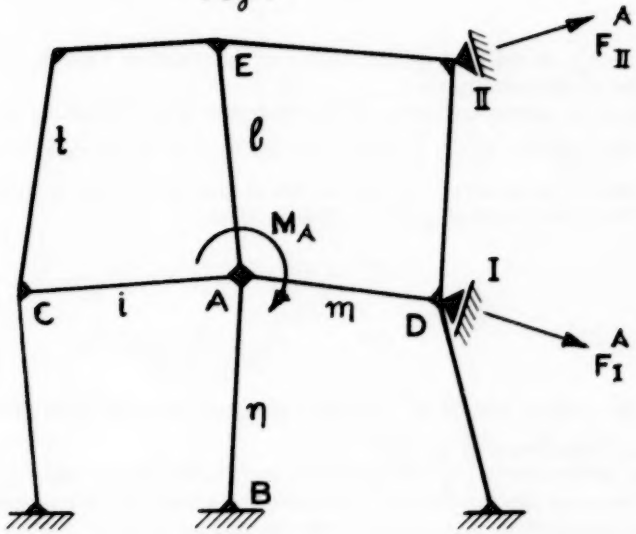

Fig. 8

where

$$C_N = C'_N \zeta'_M \quad C_T = C'_T \zeta'_M \quad d_T = d'_T \zeta'_M$$

Calculation of the Restraint Factors

6.—Let us consider the frame of the fig. (8) in which all the joints remain fixed against rotation except the joint A on which a moment $\bar{M}_A$ acts. All the joints of the frame remain linearly undisplaced because of the imaginary roller restraints I, II, III, ... The rotation of the joint A causes fixing forces F_I^A, F_{II}^A, F_{III}^A, ... at these imaginary rollers which we seek to calculate.

In the following we give the calculation of the fixing force F_I^A. The calculation of the remaining forces F_{II}^A, F_{III}^A, ... is carried out in the same way as for the F_I^A.

If we isolate the member i by two cuts near the joints A and C and we apply on the end sections externally, the developed internal forces M, N, Q (Fig. 9), the member i is in equilibrium under the action of these forces.

The distributed moment M_{Ai}^A and the carry over moment M_{Ci}^A are calculated as functions of the as follows

$$\begin{aligned} M_{Ai}^A &= - K_{Ai} \bar{M}_A \\ M_{Ci}^A &= - \mu_i K_{Ai} \bar{M}_A \end{aligned} \qquad (5)$$

where K_{Ai} is the distribution factor of the member i and μ_i is the carry over factor of the member i.

As it is known the factor μ_i in a member with a constant moment of inertia has two values: $\mu_i = +\frac{1}{2}$ and $\mu_i = 0$, the first if the member i has its end on the joint C encastered and the second if this end is free to rotate.

The shear force is given by the equation

$$Q_i^A = Q_{Ai}^A = Q_{Ci}^A = \frac{M_{Ai}^A + M_{Ci}^A}{l_i}$$

or

$$Q_i^A = - \bar{M}_A \frac{K_{Ai}\,(1+\mu_i)}{l_i} \qquad (6)$$

The normal forces N_i^A are not calculated here because they do not appear in the calculation of the F_I^A.

If we isolate any of the members t of the frame not meeting the joint A and because the joints which it meets are assumed not to rotate, this member is in equilibrium if it is acted upon its end sections by the normal force N_t^A The shear forces Q_t^A are zero.

If by cuts close to the joints we isolate all the members of the frame and on the sections of the joints we apply the N^A, Q^A, F^A as external forces, then each joint is in equilibrium under the action of these forces. As long as the systems of forces N^A, Q^A and F^A on each joint are in equilibrium, then

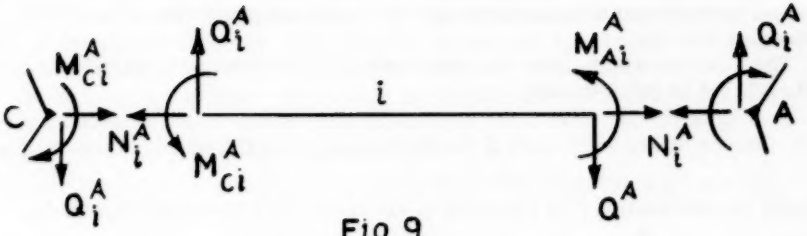

Fig. 9

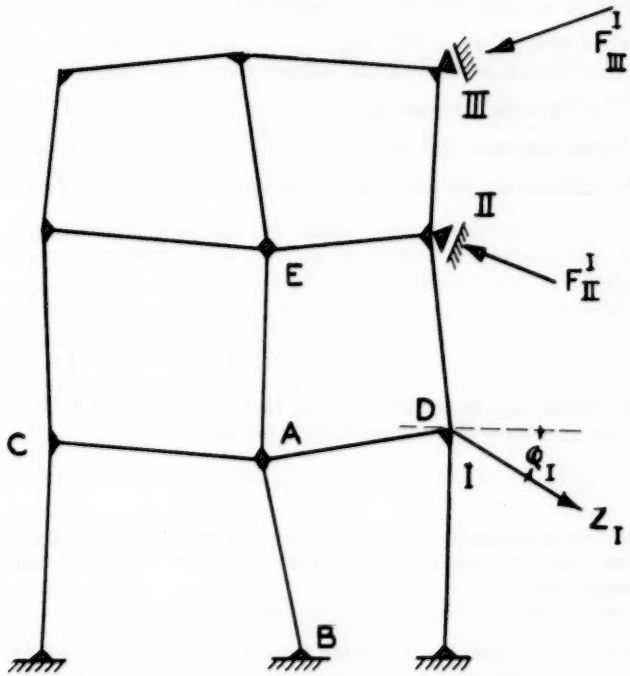

Fig. 10

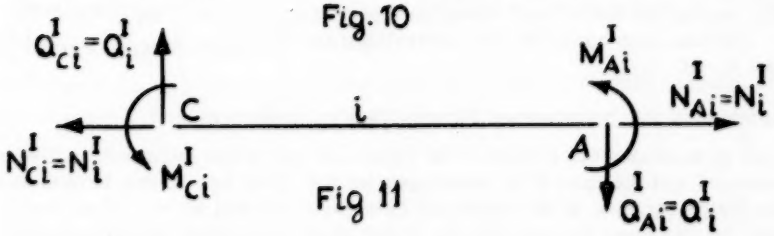

Fig 11

evidently they are in equilibrium if taken as a whole on all the joints.

In order to calculate the fixing force F_I^A we express the equilibrium of that total system of forces by applying the principle of virtual works for the virtual geometrical displacements of the joints and members of the frame belonging to (πI).

The equation which gives the equilibrium of this total system of forces Q^A, N^A, F^A is the following

$$F_I^A \; \delta_I \; \cos\varphi_I - \Sigma Q_i^A \; \delta_{iI} = 0 \qquad (7)$$

where the summation $\sum_i$ is extended to all members i meeting the joint A.

The forces N_i^A do not appear in the equation (7) because being equal and opposite in couples give works cancelling each other. For the same reason the works of the forces F_{II}^A, F_{III}^A, ... are zero, because the directions of the linear displacements of the joints, where the imaginary rollers II, III, ... are placed, are perpendicular to the directions of the F_{II}^A, F_{III}^A, ...

If in the equation (7)[2] we replace Q_i^A by their values from the equation (6) and we express δ_{II} as functions of δ_I, then we can find F_I^A as follows

$$F_I^A = - \bar{M}_A \; \sigma_{AI} \qquad (8)$$

where

$$\sigma_{AI} = \frac{1}{\cos\varphi_I} \; \sum_i \frac{c_{iI} \; K_{Ai} \; (1+\mu_i)}{\ell_i} \qquad (9)$$

The factor G_{Ai} is the restraining factor of the joint A of the group I.

In the case $\phi_I = 0^\circ$ the equation (9) becomes

$$\sigma_{Ai} = \sum_i \frac{c_{iI} \; K_{Ai} \; (1+\mu_i)}{\ell_i} \qquad (10)$$

The fixing force F_I^A is positive when parallel and of the same direction with the positive displacement of the joint from which the imaginary roller has been removed.

In the same way we can calculate the restraining factors of the group I.

In order to calculate the fixing forces F_{II}^A, F_{III}^A . . . . we express the equilibrium of the above system of the forces and we apply the principle of the virtual works for the virtual linear infinite displacement of the (π^{II}), (π^{III}), ... We can thus calculate the restraining factors G_{AII}, $G_{A_{III}}$, ... of the groups II, III, ...

Calculation of the Distribution Factors Due to Displacement

7.—Let us consider the frame of the figure 10 where the imaginary roller I is removed and the joint D is acted upon by the force Z_I. In the deformation of the frame because of the action of Z_I we assume that all the joints do not rotate. In this case the movements of the joints are linear displacements only.

2. The negative sign before the second part of the equation (7) is due to the fact that the positive shear with a positive displacement give negative work.

Because of the relative displacements δ_{iI} of the members of the frame, moments M_{Ai} . . . . . are developed at the end sections of the members. The internal forces of any member i are as if the member had its ends locked against rotation but translated relatively to each other in direction perpendicular to the original axis of itself equal to an amount δ_{iI}. In the following we give calculation of the moments M_{Ai} as functions of the force Z_I.

We isolate the member by two cuts at its ends and on its end sections we apply externally the developed internal forces (Fig. 11). The member is in equilibrium under the action of these external forces. The shear forces Q^I_{Ai}, and Q^I_{Ci}, can be calculated as functions of the moments M^I_{Ai} and M^I_{Ci} by the equation

$$Q^I_i = Q^I_{Ai} = Q^I_{Ci} = \frac{M^I_{Ai} + M^I_{Ci}}{\ell_i} \qquad (11)$$

By cuts at their ends we isolate all the members and on their end sections we apply Q^I_i , N^I_i and Z_I and the fixing forces F^I_{II}, F^I_{III}, . . . which are caused at the rest of the imaginary roller restraints II, III, ... by the Z_I.

The forces Q, N, Z and F the acting on each joint constitute systems in equilibrium. Consequently as a whole taken they constitute a system in equilibrium.

We can express the equilibrium of the whole system of these forces by applying the principle of virtual works taking as virtual displacements of the joints the real ones which have been caused on the locked joints by removal of the imaginary roller I and by the action on it of the Z_I. The relations of the elastic movements of the joints are the same as with (π^I).

The equation expressing the equilibrium of the above system of forces with the aid of the principle of virtual works is

$$Z_I \delta_I \cos\varphi_I - \Sigma c_{iI} \cdot \vartheta_I \cdot Q^I_i = 0 \qquad (12)$$

where δ_I is the elastic displacement of the joint on which the Z_I acts.

Q^I_i is the constant shear force of the member

C_{iI} a constant parameter of the member i given by the diagram of virtual displacements

i is summation extended to all members of the group I the forces N^I_i . . . . . and F^I_{II}, F^I_{III}, do not appear in the equation (12) for the same reasons as in para. 6.

The shear force Q^I_i can be calculated with the aid of the equation (11) as functions of the moments M^I_{Ai} and M^I_{Ci}.

The moments M^I_{Ai} and M^I_{Ci} of any member i with a constant moment of inertia J_i are given by the following equations

$$M^I_{Ai} = u_{Ai} \frac{\delta_{iI}}{\ell_i}$$

$$M^I_{Ci} = u_{Ci} \frac{\delta_{iI}}{\ell_i} \qquad (13)$$

where u_{Ai} and u_{Ci} are constants which depend on the geometrical characteristics of the member which is connected with joints A and C.

We call the constants u_{Ai} stiffness factors of the members.

For a member with its both ends encastered we have

$$u_{Ai} = u_{Ci} = \frac{6EJ_i}{\ell_i} \tag{14}$$

For the case the end A is encastered and the end C pin-joined

$$u_{Ai} = \frac{3EJ_i}{\ell_i} \qquad u_{Ci} = 0 \tag{15}$$

If in the equation (11) we replace M^I_{Ai} and M^I_{Ci} by their values from the equations (13) we obtain

$$Q^1_i = \delta_I \frac{(u_{Ai} + u_{Ci})\, c_{iI}}{\ell_i} \tag{16}$$

If in the equation (12) we replace Q^I_i by its value from the equation (16) and solve it we obtain

$$\delta_I = Z_I \frac{\cos\varphi_I}{\sum_I \frac{c^2_{iI}}{\ell^2_i}(u_{Ai} + u_{Ci})} \tag{17}$$

where the summation $\sum_I$ extends to all members i of the group I.

After the calculation of the elastic displacement δ_I we calculate the moment $M^I_{K\nu}$ at the end section of any member ν meeting the joint E by replacing δ_I at the equation (13) by its value from the equation (17). Then

$$M^I_{K\nu} = q^I_{K\nu} Z_I \tag{18}$$ where

$$q_{K\nu} = \frac{\frac{c_{\nu I}}{\ell_\nu} u_{K\nu}}{\sum \frac{c^2_{iI}}{\ell^2_i}(u_{Ai} + u_{Ci})} \cos\varphi_I \tag{19}$$

The factors $q^I_{K\nu}$ are called distributions factors due to displacement.

In the case of an angle $\varphi_I = 0$ the equation (19) becomes

$$q_{K\nu} = \frac{\frac{c_{\nu I}}{\ell_\nu} u_{K\nu}}{\sum_i \frac{c^2_{iI}}{\ell^2_i}(u_{Ai} + u_{Ci})} \tag{20}$$

It is evident that if we multiply the distribution factors due to displacement of all the members of the group I by the corresponding fractions $\frac{c_\nu}{\ell_\nu}$ and we add we obtain a sum equal to unity

$$\sum_\nu q^I_{K\nu} \frac{c_{\nu I}}{\ell_\nu} = 1 \tag{21}$$

where the summation $\sum_I$ extends to all members of the group I. In the same way we can calculate the distribution factors due to displacement of the groups II, III, ...

Calculation of the Shear Carry Over Factors

8.—When we remove the imaginary roller restraint I and on the joint, where this roller I is removed, we apply the force Z_I, then on the remaining rollers II, III . . . . forces F^I_{II}, F^I_{II}, ... (Fig. 9) are caused, whose seek the calculation as functions of Z_I.

In the following we give the calculation of the force F^I_{II}.

For the calculation of the F^I_{II}, we express the equilibrium of the total system of the forces Q^I, N^I, Z_I and F^I, which have been developed at the stage of para. 7, by application of the principle of virtual works where the virtual geometrical displacements of the joints and members of the frame are those of the (πII).

The equation which expresses the equilibrium of the above system forces and resulting from the application of the principle of virtual works is the following

$$F^I_{II} \cos \varphi_{II} \, \delta_{II} - \sum_i \delta_{II} \cdot c_{iII} \, Q^I_i = 0 \qquad (22)$$

where δ_{II} is the virtual displacement of the joint with the roller II.

C_{iII} the constant referring to the member i of the group II

Q^I_i the constant shear force of any member i during the stage of the para. 7.

$\sum_i$ summation extended to all members i of the group I.

If we solve the equation (22) we obtain

$$F^I_{II} = \frac{\sum_i c_{iII} \, Q^I_i}{\cos \varphi_{II}} \qquad (23)$$

the shear force Q^I_i as a function of the calculated moment is

$$Q^I_i = \cos \varphi_I \cdot Z_I \cdot \frac{\varphi^I_{Ai} + \varphi^I_{ci}}{\ell_i} \qquad (24)$$

If we replace Q^I_i from (24) to the equation (23) we obtain

$$F^I_{II} = m^I_{II} \cdot Z_I \qquad (25) \text{ where}$$

$$m^I_{II} = \frac{\cos \varphi_I}{\cos \varphi_{II}} \sum_i \frac{c_{iII}}{\ell_i} \left(\varphi^I_{Ai} + \varphi^I_{ci} \right) \qquad (26)$$

For the case of angles $\phi_I = 0$ and $\phi_I = 0$ (26) becomes

$$m^I_{II} = \sum \frac{c_{iII}}{\ell_i} \left(\varphi^I_{Ai} + \varphi^I_{ci} \right) \qquad (27)$$

The factor m^I_{II} is called shear carry over factor

The calculation of the forces F^{I}_{III}, ... is carried out in the same way by the application of the principle of virtual works with virtual displacements those of the (η^{III}) . . . . . .

Rectangular Frames

9.—The above restraining factors, distribution factor due to displacement and shear carry over factors take on a simple form in the case of rectangular frames.

In drawing up the virtual displacement diagram we consider that the section of the frame above the imaginary roller which has been removed moves without any elastic deformations (Fig. 12).

In the case of rectangular frames the displacement δ_{iI}, δ_{iII} are all equal in the columns and all equal to zero in the beams.

Restraining Factors

10.—Applying the principle of virtual works, as above, we obtain the following restraining factors (Fig. 13)

$$F^{A}_{III} = -\ (M_A)\ \sigma_{AIII} \ldots\ldots \qquad (28)$$

where

$$\sigma_{AIII} = \frac{K_{Ai}\ (1+\mu_i)}{\ell_i}$$

For the calculation of forces F^{A}_{IV} and in the application of the principle of virtual works, we must bear in mind that the calculated force F^{A}_{III} performs work, too, due to the displacement of the joint on which it acts.

$$F^{A}_{IV} + F^{A}_{III} - (-M_A)\ \frac{K_{At}\ (1+\mu_t)}{\ell_t} = 0 \ldots\ldots \qquad (29)$$

$$F^{A}_{IV} = -\ (M_A)\ \sigma_{AIV} \ldots\ldots\ldots\ldots \qquad (30)$$

where

$$\sigma_{AII} = \frac{K_{At}\ (1+\mu_t)}{\ell_t} - \frac{K_{Ai}\ (1+\mu_i)}{\ell_i}$$

$$F^{A}_{V} = -\ (M_A)\ \sigma_{AV} \qquad (31)$$

where

$$\sigma_{AV} = -\ \frac{K_{At}\ (1+\mu_t)}{\ell_t}$$

Distribution Factors Due to Displacement

11.—The distribution factors due to displacement takes the following simpler form (Fig. 12):

$$q^{II}_{K\rho} = h_{II}\ \frac{u_{K\rho}}{\Sigma (u_{Ai} + u_{Ci})} \ldots\ldots\ldots \qquad (32)$$

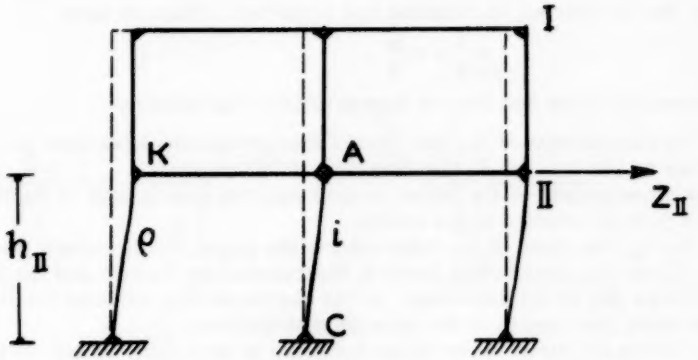

Fig. 12

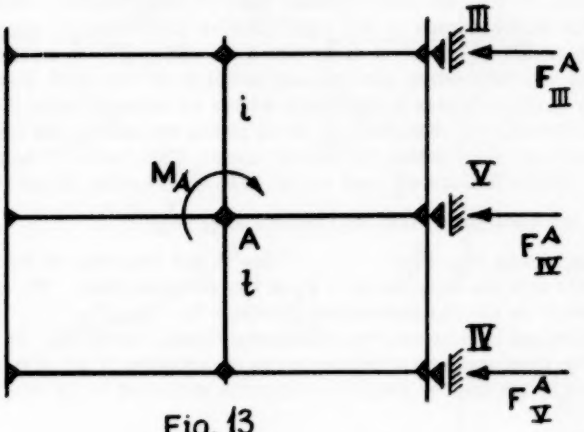

Fig. 13

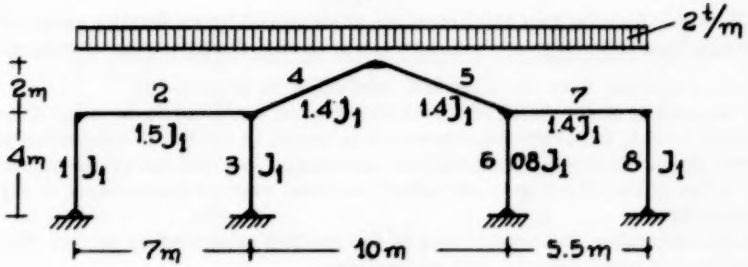

Fig. 14

Shear Carry Over Factor

12.—In the case of rectangular frames of (Fig. 12) the shear carry over factor, for all stories, is constant and equal to i. Thus we have:

$$m_{II}^{I} = m_{III}^{II} = \ldots\ldots\ldots = -1$$

Arrangement of the Successive Approximation Calculations

13.—The calculations of the successive approximations have been arranged in tables as can be seen in the examples of this paper.

The arrangement of the tables is such that the distribution of the moments at each joint is effected in the tables.

In the top few lines of the table we list the joints, the members meeting at these joints, the distribution factors, the restraining factors and the distribution factors due to displacement. In the corresponding columns for the joints we set down the results of the moment distributions.

Following the distribution of the moments at each joint we add up the moments at the end section of each member and determine the moments resulting from the distribution which we write down in a rectangle, in the relative column, so that the distributions may be continuously checked.

The last columns to the right give us the fixing forces F_I, F_{II} . . . . After the moment M has been distributed at each joint we multiply the moment - M by the corresponding restraining factor G for the joint and write down the results in the columns to the right where we already have the fixing forces.

Following the distribution at all joints we add up the figures in the F_I column and thus obtain the fixing force. This (with a changed sign) we multiply first by factors q_{kl}^{I} and write down the results in the corresponding column for each joint, then by factors M_{II}^{I}, M_{III}^{I} . . . . . , to determine the fixing forces F_{II}^{I}, F_{III}^{I} . . . . . due to the removal of the support on which F_I acts and the exertion of - F_I on the relative joint. We write down these products in the corresponding columns for F_{II}, F_{III}; . . . . The same process is repeated in turn for the remaining fixing forces F_{II}, F_{III} . . . . .

The final moments following the completion of all distributions, are obtained by adding the relative moments enclosed in rectangles.

Conclusion

14.—It is our opinion that the new method proposed by us for the calculation of frames by successive approximations presents the following advantages:

1) It is quicker than the classical method of H. Cross, and

2) depending of the form of the structure, the material to be used in construction (steel, reinforced concrete) it is possible for us to determine the number of successive approximations necessary, so that the grade of accuracy in the calculations may not affect the final section dimensions of structure members.

In the examples, the advantages of the method proposed by us, for the calculation of rigid frames, are made clear.

Application

1) The bending moment diagrams of the following frame are to be drawn (Fig. 14)

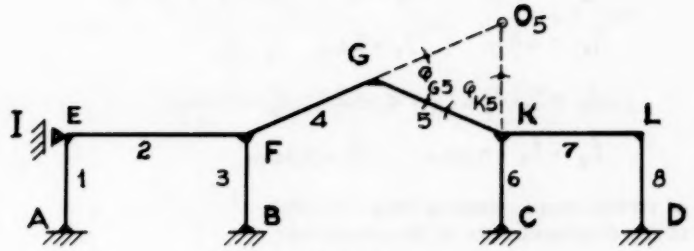

Fig. 15

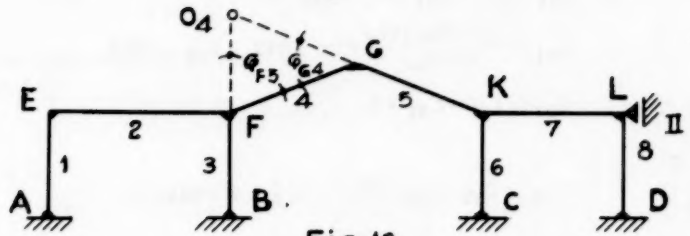

Fig. 16

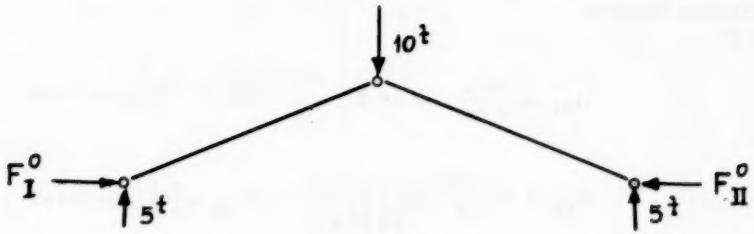

Fig. 17

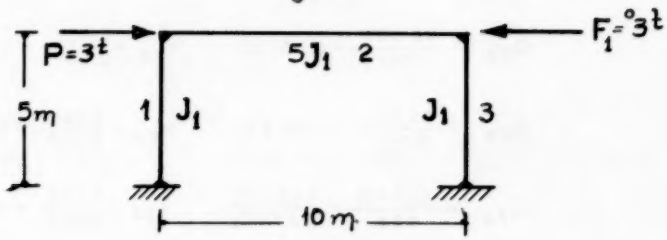

Fig. 18

Data $q = 2.0\,t/m$

$$J_2 = 1.5J_1 \qquad J_3 = J_1 \qquad J_4 = J_5 = 1.4J_1$$

$$J_6 = 0.8J_1 \qquad J_7 = 1.4J_1 \qquad J_8 = J_1$$

$$l_1 = l_3 = l_6 = l_8 = 4.00m \qquad l_2 = 7.00m$$

$$l_4 = l_5 = 5.38m \qquad l_7 = 5.50m$$

Diagrams of virtual displacements (Fig. 15, 16)
Constants of the displacements of the members:
a) group I

$$c_{1I} = 0 \qquad c_{2I} = 0 \qquad c_{3I} = 0$$

$$c_{4I} = \frac{\sin(\varphi_{F4} + \varphi_{c4})}{\sin \varphi_{c4}} = -1.342 \qquad c_{5I} = 1.342$$

$$c_{6I} = c_{7I} = c_{8I} = 0$$

b) group II

$$c_{1II} = c_{2II} = c_{3II} = 0 \qquad c_{4II} = +1.342$$

$$c_{5II} = -1.342 \qquad c_{6II} = 1.00 \qquad c_{7II} = 0 \qquad c_{8II} = 1.00$$

Distribution factors:
Joint E:

$$a_{E1} = \frac{4EJ_1}{4.00} = 1.00\,EJ_1 \qquad K_{E1} = \frac{1.00}{1.857} = 0.538$$

$$a_{E2} = \frac{4E \cdot 1.5J_1}{7.00} = \frac{0.857\,EJ_1}{1.857\,EJ_1} \qquad K_{E2} = \frac{0.857}{1.857} = 0.462$$

Joint F:

$$a_{F2} = \frac{4E\,1.5J_1}{7.00} = 0.857EJ_1 \qquad K_{F2} = \frac{0.857}{2.897} = 0.295$$

$$a_{F3} = \frac{4EJ_1}{4.00} = 1.00\,EJ_1 \qquad K_{F3} = \frac{1.000}{2.897} = 0.345$$

$$a_{F4} = \frac{4E1.4J_1}{5.38} = \frac{1.04\,EJ_1}{2.897EJ_1} \qquad K_{F4} = \frac{1.040}{2.897} = 0.360$$

Joint G:

$$K_{G4} = 0.500$$

$$K_{G6} = 0.500$$

Joint K:

$$a_{K5} = \frac{4E.1.47J_1}{5.38} = 1040\ EJ_1 \qquad K_{K5} = \frac{1.040}{2.860} = 0.364$$

$$a_{K6} = \frac{4E.08J_1}{4.00} = 0.800\ EJ_1 \qquad K_{K6} = \frac{0.800}{2.860} = 0.280$$

$$a_{K7} = \frac{4E.1.47J_1}{5.50} = \frac{1.020\ EJ_1}{2860\ EJ_1} \qquad K_{K7} = \frac{1.020}{2.860} = 0.356$$

Joint L:

$$a_{L7} = \frac{4E.1.47J_1}{5.50} = 1.020 EJ_1 \qquad K_{L7} = \frac{1.020}{2.020} = 0.505$$

$$a_{L8} = \frac{4E\,J_1}{4.00} = \frac{1.000\,EJ_1}{2.020\,EJ_1} \qquad K_{L8} = \frac{1.000}{2.020} = 0.495$$

Restraint factors of the joints:
a) group I

$$\sigma_{EI} = \frac{3 \times 0538}{2 \times 4.00} = +0.202$$

$$\sigma_{FI} = \frac{3 \times 0.345}{2 \times 4.00} - \frac{3 \times 1.342 \times 0.360}{2 \times 538} = -0.005$$

$$\sigma_{GI} = -\frac{3 \times 1.342 \times 0.500}{2 \times 538} + \frac{3 \times 1.342 \times 0.50}{2 \times 5.38} = 0$$

$$\sigma_{KI} = \frac{3 \times 1.342 \times 0.364}{2 \times 5.38} = 0.136 \qquad \sigma_{L_I} = 0$$

b) group II

$$\sigma_{EII} = 0 \qquad \sigma_{FII} = \frac{3 \times 0.360 \times 1.342}{2 \times 5.38} = +0.135$$

$$\sigma_{GII} = 0 \qquad \sigma_{KII} = \frac{3 \times 0.364 \times 1.342}{2 \times 5.38} + \frac{3 \times 0.280 \times 1}{2 \times 4.00} = -0.031$$

$$\sigma_{LII} = \frac{3 \times 0495}{2.400} = 0.186$$

Stiffness factors:

$$u_{E1} = u_{A1} = \frac{6EJ_1}{4} = 1.5EJ_1 \qquad u_{E2} = u_{F2} = \frac{6E.1.5J_1}{700} = 1.287EJ_1$$

$$u_{F3} = u_{B3} = \frac{6EJ_1}{4} = 1.5EJ_1 \qquad u_{FA} = u_{64} = \frac{6E\ 1.47J_1}{5.38} = 1.56\ EJ_1$$

$$u_{G5} = u_{K5} = 1.56EJ_1 \qquad u_{K7} = u_{L7} = \frac{6E\ 147J_1}{5.50} = 1.53\ EJ_1$$

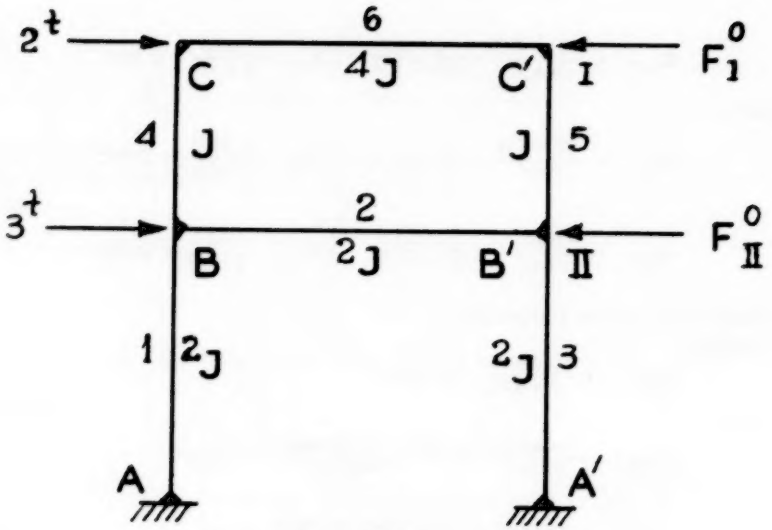

Fig. 19

TABLE I

No m	l_i	c_{iI}	c_{iII}	c_{iI}/l_i	c_{iII}/l_i	$(c_{iI}/l_i)^2$	$(c_{iII}/l_i)^2$
1	4,00	1,00	0,00	0,25	0,00	0,0625	0,000
2	7,00	0,00	0,00	0,00	0,00	0,000	0,000
3	4,00	1,00	0,00	0,25	0,00	0,0625	0,000
4	5,38	-1,342	+1,342	-0,25	+0,25	0,0625	0,0625
5	5,38	+1,342	-1,342	+0,25	-0,25	0,0625	0,0625
6	4,00	0,00	1,00	0,00	+0,25	0,000	0,0625
7	5,50	0,00	0,00	0,00	0,00	0,000	0,000
8	4,00	0,00	1,00	0,00	+0,25	0,000	0,0625

$$u_{K6} = u_{G6} = \frac{6E\,08J_1}{4.00} = 1.2EJ_1 \quad u_{L8}\,u_{D8} = \frac{6EJ_1}{4.00} = 1.5EJ_1$$

By aid of Table I we calculate the following factors:
Distribution factors due to displacements:
a) group I

$$\Sigma\,(\,) = 0.0625 \times 2\,(1.50+1.50+1.56+1.56)\,EJ_1 = 0.765\,EJ_1$$

$$q^{I}_{E1} = q^{I}_{A1} = \frac{0.250 \times 1.5}{0.765} = 0.490$$

$$q^{I}_{F3} = q^{I}_{B3} = \frac{0.250 \times 1.5}{0.765} = 0.490 \qquad q^{I}_{E2} = q^{I}_{F2} = 0$$

$$q^{I}_{F4} = q^{I}_{G4} = -\frac{0.250 \times 1.56}{0.765} = -0.510$$

$$q^{I}_{G5} = q^{I}_{K5} = 0.510 \qquad q^{I}_{K6} = q^{I}_{C6} = 0$$

$$q^{I}_{K7} = q^{I}_{L7} = 0 \qquad q^{I}_{L8} = q^{I}_{D8} = 0$$

b) group II

$$\Sigma\,(\,) = 0.0625 \times 2\,(1.56+1.56+1.20+1.20)\,EJ_1 = 0727\,EJ_1$$

$$q^{II}_{E1} = q^{II}_{A1} = 0 \qquad q^{II}_{E2} = q^{II}_{F2} = 0 \qquad q^{II}_{F3} = q^{II}_{B3} = 0$$

$$q^{II}_{F4} = q^{II}_{G4} = \frac{0.250 \times 1.56}{0.727} = 0.536 \qquad q^{II}_{G5} = q^{II}_{K5} = 0.536$$

$$q^{II}_{K6} = q^{II}_{C6} = \frac{0.250 \times 1.20}{0.727} = 0.413 \qquad q^{II}_{K7} = q^{II}_{L7} = 0$$

$$q^{II}_{L8} = q^{II}_{D8} = \frac{0.250 \times 1.50}{0.727} = 0.515$$

Check:

$$\Sigma \frac{c_{iI}}{l_i}\, q^{II}_{kl} = 0.25 \times 2 \quad (0.490+0.490+0.510+0.510) = 1$$

$$\Sigma \frac{c_{iII}}{l_i}\, q^{II}_{Ki} = 0.25 \times 2 \quad (0.536+0.536+0.413+0.515) = 1$$

TABLE II

	A	B	E		F			G		K			L		C	D	F_I	F_{II}
	1	3	1	2	2	3	4	4	5	5	6	7	7	8	6	8	m_I^I	m_{II}^I
K_{AI}			0.538	0.462	0,295	0,345	0.360	0,500	0.500	0,364	0,280	0,356	0,505	0,495	—	—	—	-0,510
δ_{AI}			+ 0,202			- 0,005					+ 0,136			—	—	—	m_I^{II}	m_{II}^{II}
δ_{AII}						+ 0,135					- 0,031			+ 0,186			- 0,536	
q_{AI}^I	0.490	+0,490	+0,490			+0,490	-0,510	-0,510	0.510	0,510								
q_{AI}^{II}							+0,536	0,536	-0,536	-0,536	0,413			0,515	0,413	0,515		
				8,160	-8,160		4,160	-4,160	4,160	-4,160		5,040	-5,040				12,500	-12,500
	-2,195		-4,390	-3,770	-1,885												-1,650	
		1,015	-4,390	4,390	1,735	2,030	2,120	1,060									-0,029	0,800
				0,868	-8,310	2,030	6,280	-0,530	-0,530	-0,265							—	—
							-0,265	-3,630	3,630	-0,224	-0,172	-0,219	-0,110		-0,086		-0,084	0,019
									-0,112	-4,649	-0,172	4,821	2,600	2,550		1,275	—	0,956
												1,300	-2,550	2,550				
	-5,270	-5270	-5,270			-5,270	5,480	5,480	-5,480	-5,480							10,737	5,480
							2,810	2,810	-2,810	-2,810	2,160			2,700	2,160	2,700	-2,810	-5,245
	1,185		2,370	2,032	1,016												0,890	—
		-0,650	-2,900	2,900	-1,111	-1,300	-1,360	-0,680									0,019	-0,507
				-0,555	-0,095	-6,570	6,665	0,396	0,396	0,198							—	—
							0,198	8,006	-8,006	1,686	1,296	1,650	0,825		0,648		0,630	-0,140
									0,843	-6,406	3,456	2,950	-1,780	-1,745		-0,872	—	-0,655
	0,623	0,623	0,623			0,623	-0,648	-0,648	0,648	0,648		-0,890	-0,955	0,955			-1,271	-0,648
							1,045	1,045	-1,045	-1,045	0,805			1,000	0,805	1,000	-1,045	-1,950
	-0,018		-0,037	-0,031	-0,016												-0,014	—
		-0,208	0,586	-0,586	-0,355	-0,415	-0,432	-0,216									0,006	-0,163
				-0,178	-0,371	0,208	0,163	-0,313	-0,314	-0,157							—	—

| | | |

Shear carry over factors:

$$m_{II}^{I} = +0.250\ (-0.510-0.510)-0.250(+0.510+0.510)=-0.510$$

$$m_{I}^{II} = 0.250\ (0.536+0.536)+0.250(-0.536-0.536)=-0.536$$

Active moments:

$$\bar{M}_{E2}=-\bar{M}_{F2} = \frac{2\times7.00^2}{12} = 8.16\ t.m.$$

$$\bar{M}_{F4}=-\bar{M}_{G4} = \frac{2\times5.00^2}{12} = 4.16\ t.m.$$

$$\bar{M}_{G5}=-\bar{M}_{K5} = \frac{2\times5.00^2}{12} = 4.16\ t.m$$

$$\bar{M}_{K7}=-\bar{M}_{L7} = \frac{2\times5.50^2}{12} = 5.04\ t.m$$

Fixing forces (Fig. 17): $F_I^0 = +12.5^t$ $F_{II}^0 = -12.5^t$

The successive approximations and the arrangement of the operations is tabulated in the Table II. 2) The bending moment diagrams of the following frame are to be drawn (fig. 18)

Distribution factors

Joint C:

$$a_{C1} = \frac{4EJ_1}{5.00} = 0.800\,EJ_1 \qquad K_{C1} = \frac{0.8}{2.8} = 0.286$$

$$a_{C2} = \frac{4E\times5J_1}{10.00} = \frac{2.000EJ_1}{2.800EJ_1} \qquad K_{C2} = \frac{2.0}{2.8} = 0.714$$

Joint:

$$K_{D2} = 0.714$$

$$K_{D3} = 0.286$$

Restrain factors of the joints

$$G_C = \frac{3\times0.286}{2\times50} = 0.0858 \qquad G_D = 0.0858$$

Stiffness factors

$$u_{C1} = u_{A1} = u_{D3} = u_{B3} = \frac{6EJ}{30} = 1.20\,EJ$$

Distribution factors due to displacements

$$q_{A1} = q_{C1} = q_{B1} = q_{D1} = 500\times\frac{1.20}{4\times1.20} = 1.250$$

$$F^0 = -300^t$$

The successive approximations and the arrangement of the operations is tabulated in the table III. 3) The bending moment diagrams of the following frame are to be drawn (Fig. 19).

TABLE III

	A	C		D		B	F
	1	1	2	2	3	3	
K_{Ai}	—	0,286	0,714	0,714	0,286	—	
σ_A	—	+ 0,0858		+ 0,0858		—	
q_{Ai}	1,250	1.250	—	—	1.250	1,250	
	+3,750	+ 3,750			+ 3,750	+3,750	-3,000
	-0,536	- 1,072	-2,678	- 1,339			-0,322
		+ 2,678	-2,678	- 1,720	-0,691	-0,345	-0,207
			-0,860	-3,059	+3,059		-0,529
	+0,661	+0,661			+0,661	+0,661	
	+0,029	+0,057	+0,142	+0,071			+0,017
		+0,718	-0,718	-0,522	-0,210	-0,105	-0,063
			-0,261	-0,451	+0,451		-0,046
	+0,057	+0,057			+0,057	+0,057	
	+0,029	+0,058	+0,146	+0,073			+0,017
		+0,115	-0,115	-0,093	-0,037	-0,019	-0,011
			-0,046	-0,020	+0,020		+0,006
	-0,008	-0,008			-0,008	-0,008	
	+0,007	+0,015	+0,039	+0,019			+0,005
		+0,007	-0,007	-0,008	-0,003	-0,002	-0,001
			-0,004	+0,011	-0,011		+0,004
	-0,005	-0,005			-0,005	-0,005	
	+0,002	+ 0,003	+0,006	+0,003			—
		-0,002	+0,002	+ 0,001	+0,001		—
				+ 0, 004	-0,004		
M_{Ki}	+3,986	+3,516	-3,516	- 3.515	+ 3.515	+3,984	

Distribution factors

Joint C:

$$a_{C4} = \frac{4EJ}{400} = 100EJ \qquad K_{C4} = \frac{100}{300} = 0.333$$

$$a_{C5} = \frac{4E\times 4J}{800} = \frac{2.0\,0EJ}{3.0\,0EJ} \qquad K_{C5} = \frac{200}{300} = 0.667$$

Joint C':

$$K_{C'3} = 0.667$$

$$K_{C'6} = 0.333$$

Joint B:

$$a_{B4} = \frac{4EJ}{400} = 1.00\,EJ \qquad K_{B4} = \frac{1.000}{3.333} = 0.300$$

$$a_{B2} = \frac{4E\times 2J}{8} = 1.00\,EJ \qquad K_{B2} = \frac{1.000}{3.333} = 0.300$$

$$a_{B1} = \frac{4E\times 2J}{600} = \frac{1.333EJ}{3.333EJ} \qquad K_{B1} = \frac{1.333}{3.333} = 0.400$$

Joint B':

$$K_{B'6} = 0.300$$

$$K_{B'2} = 0.300$$

$$K_{B'3} = 0.400$$

Restrain factors of the joints

$$\sigma_{CI} = \frac{3}{2}\times\frac{0.333}{400} = 0.125 \qquad \sigma_{C'I} = 0.125$$

$$\sigma_{BI} = \frac{3}{2}\times\frac{0.300}{400} = 0.1125 \qquad \sigma_{B'I} = 0.1125$$

$$\sigma_{CII} = -\frac{3}{2}\times\frac{0.333}{} = -0.125 \qquad \sigma_{C'II} = -0.125$$

$$\sigma_{BII} = \frac{3}{2}\left(\frac{0.400}{6.00} - \frac{0.300}{400}\right) = -0.0125 \qquad \sigma'_{BII} = -0.0125$$

Stiffness factors

$$u_{C4} = u_{B4} = u_{C'6} = u_{B'C} = \frac{6EJ}{4.00} = 1.50\,EJ$$

$$u_{B1} = u_{A1} = u_{B'3} = u_{A'3} = \frac{6E2J}{6} = 2.00\,EJ$$

Distribution factors due to displacements

$$q^I_{C4} = q^I_{B4} = q^I_{C'6} = q^I_{B'6} = 4.00\,\frac{1.50}{4\times 1.50} = 1.00$$

$$\varphi^{II}_{B1} = \varphi^{II}_{A1} = \varphi^{II}_{B'3} = \varphi^{II}_{A'3} = 600 \frac{2EJ}{4 \times 2EJ} = 150$$

$$F^{o}_{I} = -2\overset{t}{.}00 \qquad F^{o}_{II} = -3.00^{t}$$

The successive approximations and the arrangement of the operations is tabulated in the table IV.

TABLE IV

	C		C'		B			B'			A	A'	F_I	F_{II}
	4	5	5	6	4	2	1	6	2	3	1	3		
	0,333	0,667	0,667	0,333	0,300	0,300	0,400	0,300	0,300	0,400	—	—		m_{II}^{I}
δ_{KI}	0,125		0,125		0,1125			0,1125			—	—		-1,000
δ_{KII}	- 0,125		- 0,125		-0,0125			-0,0125			—	—		
q_{Ki}^{I}	1,000			1,000	1,000			1,000						
q_{Ki}^{II}							1,500			1,500	1,500	1,500		
	+2,000			+2,000	+2,000			+2,000					-2,000	-3,000
														-2,000
							+7,500			+7,500	+7,500	+7,500		-5,000
	-0,666	-1,334	-0,667		-0,333								-0,250	+0,250
	+1,334	-1,334	-0,889	-0,444				-0,222					-0,167	+0,167
	-1,375	-0,445	-1,556	+1,556	-2,750	-2,750	-3,667		-1,375		-1,834		-1,030	+0,114
				-1,185	-1,083	-2,750	+3,833	-2,370	-2,370	-3,163		-1,581	-0,890	+0,099
						-1,185		-0,592	-3,745	+4,337				
	+2,337			+2,337	+2,337			+2,337					-2,337	-2,337
							+2,560			+2,560	+2,560	+2,560		-1,707
	-0,172	-0,345	-0,172		-0,086								-0,065	+0,065
	+0,790	-0,790	-0,654	-0,326				-0,163					-0,123	+0,123
	-0,544	-0,327	-0,826	+0,826	-1,088	-1,088	-1,450		-0,544		-0,725		-0,408	+0,045
				-0,628	+1,163	-2,273	+1,110	-1,257	-1,257	-1,676		-0,838	-0,471	+0,052
						-0,628		+0,917	-1,801	+0,884				
	+1,067			+1,067	+1,067			+1,067					-1,067	-1,067
							+1,170			+1,170	+1,170	+1,170		-0,782
	-0,065	-0,131	-0,065		-0,032								-0,024	+0,024
	+0,458	-0,458	-0,249	-0,125				-0,062					-0,047	+0,047

	-0,236	-0,124	-0,314	+0,314	-0,473	-0,473	-0,631		-0,236		-0,315		-0,177	+0,020
				-0,291	+0,562	-1,101	+0,539	-0,582	-0,582	-0,775		-0,387	-0,218	+0,024
						-0,291		+0,423	-0,818	+0,395				
	+0,466			+0,466	+0,466			+0,466					-0,466	-0,466
							+0,526			+0,526	+0,526	+0,526		-0,351
	-0,035	-0,071	-0,035		-0,017								-0,013	+0,013
	+0,195	-0,195	-0,093	-0,047				-0,023					-0,017	+0,017
	-0,102	-0,047	-0,128	+0,128	-0,205	-0,205	-0,274		-0,102		-0,137		-0,077	+0,008
				-0,130	+0,244	-0,496	+0,252	-0,260	-0,260	-0,347		-0,173	-0,098	+0,011
						-0,130		+0,183	-0,362	+0,179				
	+0,205			+0,205	+0,205			+0,205					-0,205	-0,205
	-0,020	-0,036					+0,234			+0,234	+0,234	+0,234		-0,156
	-0,020	-0,036	-0,018		-0,010								-0,007	+0,007
	+0,084	-0,084	-0,038	-0,019				-0,010					-0,007	+0,007
	-0,045	-0,019	-0,056	+0,056	-0,090	-0,090	-0,119		-0,045		-0,060		-0,034	+0,004
				-0,057	+0,105	-0,220	+0,115	-0,115	-0,115	-0,154		-0,077	-0,043	+0,005
						-0,057		+0,080	-0,160	+0,080				
	+0,091			+0,091	+0,091			+0,091					-0,091	-0,091
							+0,100			+0,100	+0,100	+0,100		-0,067
	-0,009	-0,018	-0,009		-0,004								-0,003	+0,003
	+0,037	-0,037	-0,017	-0,008				-0,004					-0,003	+0,003
			-0,026	+0,026	-0,039	-0,039	-0,052		-0,020		-0,026		-0,015	+0,002
					+0,048	-0,096	+0,048	-0,050	-0,050	-0,067		-0,033	-0,019	+0,002
								+0,037	-0,070	-0,033				
M_{ki}	+2,898	-2,898	-2,906	+2,906	+1,039	-6,936	+5,897	+1,048	-6,956	+5,908	+8,993	+9,001		

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1955

PRESIDENT

WILLIAM ROY GLIDDEN

VICE-PRESIDENTS

Term expires October, 1955:
ENOCH R. NEEDLES
MASON G. LOCKWOOD

Term expires October, 1956:
FRANK L. WEAVER
LOUIS R. HOWSON

DIRECTORS

Term expires October, 1955:
CHARLES B. MOLINEAUX
MERCEL J. SHELTON
A. A. K. BOOTH
CARL G. PAULSEN
LLOYD D. KNAPP
GLENN W. HOLCOMB
FRANCIS M. DAWSON

Term expires October, 1956:
WILLIAM S. LaLONDE, JR.
OLIVER W. HARTWELL
THOMAS C. SHEDD
SAMUEL B. MORRIS
ERNEST W. CARLTON
RAYMOND F. DAWSON

Term expires October, 1957:
JEWELL M. GARRELTS
FREDERICK H. PAULSON
GEORGE S. RICHARDSON
DON M. CORBETT
GRAHAM P. WILLOUGHBY
LAWRENCE A. ELSENER

PAST-PRESIDENTS

Members of the Board

WALTER L. HUBER
DANIEL V. TERRELL

EXECUTIVE SECRETARY
WILLIAM H. WISELY

TREASURER
CHARLES E. TROUT

ASSISTANT SECRETARY
E. L. CHANDLER

ASSISTANT TREASURER
CARLTON S. PROCTOR

PROCEEDINGS OF THE SOCIETY

HAROLD T. LARSEN
Manager of Technical Publications

DEFOREST A. MATTESON, JR.
Editor of Technical Publications

PAUL A. PARISI
Assoc. Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

SAMUEL B. MORRIS, *Chairman*

JEWELL M. GARRELTS, *Vice-Chairman*

GLENN W. HOLCOMB
OLIVER W. HARTWELL
ERNEST W. CARLTON
DON M. CORBETT